Abrupt transition in a sandpile model

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We present a fixed energy sandpile model which, by increasing the initial energy, undergoes, at the level of individual configurations, a discontinuous transition. The model is obtained by modifying the toppling procedure in the Bak-Tang-Wiesenfeld (BTW) [Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988)] rules: the energy transfer from a toppling site takes place only to neighboring sites with less energy (negative gradient constraint) and with a time ordering (asynchronous). The model is minimal in the sense that removing either of the two above-mentioned constraints (negative gradient or time ordering) the abrupt transition goes over to a continuous transition as in the usual BTW case. Therefore, the proposed model offers a unique possibility to explore at the microscopic level the basic mechanisms underlying discontinuous transitions.

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I. INTRODUCTION

Self-organized criticality (SOC) has been proposed as a universal mechanism leading to scaling laws in the dynamics of driven systems that evolute towards a critical state without the fine tuning of a control parameter [1]. Recently it has been realized [2–7] that the approach to criticality is in fact controlled by the driving force as well as by the dissipation present in the dynamics. These observations initiated an extensive search for the influence of the values of these control quantities to the critical behavior of the corresponding SOC models. As a consequence, an alternative way to study SOC models emerged, restricting the considerations in closed systems (without external driving and dissipation) obeying the same dynamical rules. In these models, the energy density ρ is exactly conserved. Since the fixed-energy sandpiles (FESs) possess simpler dynamics (without loss or addition) and are translationally invariant, they are easier to study than their SOC counterparts. Furthermore, the order parameter in this case can be easily identified: below a critical density the dynamics lead to an absorbing state characterized by the absence of activity (energy transfer processes). For densities above the critical ρ_c, the system sustains activity. Thus, the critical properties of the system can be explored defining as order parameter the density of active sites and studying its dependence on ρ. In addition the measured critical exponents in absorbing-state phase transitions can be related to avalanche exponents measured in slowly driven systems [8,9].

Our analysis will be devoted exclusively to the sandpile models, although many of our results could apply as well to other systems possessing SOC [10]. Usually the transition from an absorbing (vanishing order parameter) to an active state (order parameter different from zero) in FES models is continuous [11], allowing their classification in universality classes. On the other hand, there is a variety of physical processes that are characterized by a discontinuous transition (melting, boiling, earthquake events, etc.). It is therefore natural to ask if such systems could be described in terms of fixed-energy sandpile models. In the present work we show that a suitable modification of the toppling rules in the Bak-Tang-Wiesenfeld (BTW) model can lead to a discontinuous transition. Even more, we determine two conditions, imposed on the BTW dynamics, which are both neccessary and sufficient to obtain an abrupt (first-order) transition in the corresponding FES model: the toppling of an energetically activated site involves energy transfer processes only to less energetic neighbors and takes place sequentially in time. This observation opens the possibility to design devices with extreme sensitivity on control parameters by applying the analogous constraints.

Discontinuous transitions in SOC models have already been considered in the context of the so-called stick-slip dynamics [12] or the breakdown of disordered media [13]. However, our approach here is different as it is based entirely on slight modifications of the BTW rules. This paper is organized as follows: in Sec. II we present the dynamics and describe the critical properties of the proposed model. In addition we compare the obtained results with the critical behavior of conventional FES models. In Sec. III an intepretation (also in terms of microscopic dynamics) of the numerical results of Sec. II is given, and finally in Sec. IV we summarize our findings and give a brief outlook.

II. THE SEQUENTIAL MODIFIED BTW (SMBTW) FES SANDPILE MODEL AND ITS CRITICAL BEHAVIOR

The FES model we use is the following: we randomly depose energy on a square lattice in the form of grains. Each site can have an arbitrary number of grains. The total energy (and therefore the total number of grains) is fixed by the given energy density ρ. We denote as $z_{i,j}$ the number of grains on the site (i,j). A site is characterized as active if $z_{i,j}$ exceeds or is equal to a threshold value z_c. An active site topples and grains are transferred from this site $[(i,j)]$ to the next neighboring sites provided the corresponding $z_{i\pm1,j}$, $z_{i,j\pm1}$ are less than $z_{i,j}$. We use the same threshold value z_c as in the original BTW model ($z_c=4$). We impose, however, the con-

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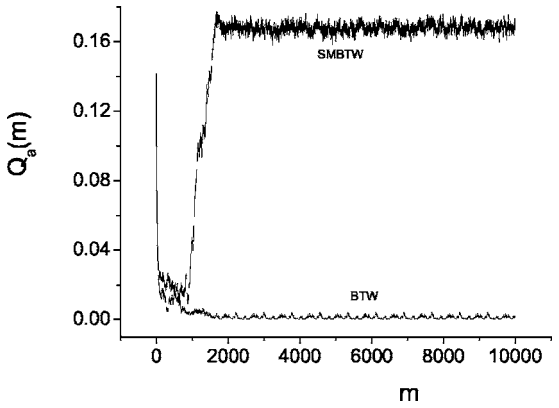

FIG. 1. The function $Q_a(m)$ for a single configuration for both the SMBTW FES model using $\rho=2.0496$ as well as the BTW FES model ($\rho=2.1151$).

straint that the toppling procedure is accomplished sequentially: the site (i,j) can topple only if the (suitably defined) previous site has already toppled. The ordering of the toppling times of the lattice sites can be chosen randomly with the constraint that the entire lattice is covered once. This sequence does not need to be the same in the following sweeps.

We have explored the dynamics of the system using an $L\times L$ square lattice and periodic boundary conditions. As usually is the case for the FES models, the control parameter is the density of grains ρ which is a conserved quantity. As initial conditions, we have used a "microcanonical" ensemble of N configurations obtained by placing randomly, with a uniform distribution, ρL^2 grains on the lattice. As an observable to characterize the evolution of the system within the ensemble, we use the mean density of active sites $Q_a(m)$ at time m. A stationary state is described through the corresponding value of $Q=Q_a(\infty)$. For most of our numerical simulations, we have used $N=1000$ and $L=120$. We observe that for low densities ($\rho<2.0495$) all initial configurations lead to the absorbing state $Q=0$. For $\rho>2.0495$ (in steps 10^{-4} for $100\leqslant L\leqslant 316$) there is a percentage w of configurations that lead to $Q=0$ and the remaining configurations lead to $Q\approx 0.17$. In fact, in the latter case, $Q_a(m)$ approaches, in the asymptotic limit (for $m\gg 1$), a stationary state described by random fluctuations of amplitude 0.01 around the value 0.17. In Fig. 1 we show the function $Q_a(m)$ for one such configuration at $\rho=2.0496$. For comparison, we display in the same plot the corresponding evolution for a typical configuration with density just above the critical one ($\rho=2.1151$) in the usual BTW FES model.

It is clearly seen that there is an energy gap between the stationary states of the system in the case of the SMBTW FES model. No such gap can be observed in BTW FES dynamics at this scale. The value of Q just above the critical density is two orders of magnitude smaller in the BTW than in the SMBTW case. It is then natural to assign two phases to the SMBTW system: phase A is characterized by $Q=0$ (absorbing state) while phase B corresponds to $Q\approx 0.17$. Thus, the value $\rho_c=2.0495$ represents the critical density of the system for the given lattice size and set of initial configu-

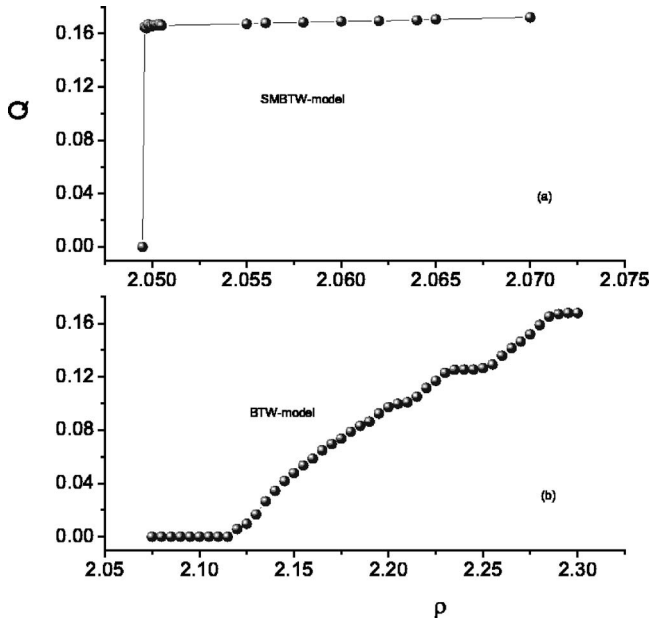

FIG. 2. (a) The order parameter Q as a function of the control parameter ρ of the SMBTW FES model. For $\rho>\rho_{c,\mathrm{SMBTW}}$, we use, in the ensemble averaging, only configurations leading to $Q\neq 0$. (b) The corresponding plot for the order parameter Q of the FES BTW model. As in (a), we use in the ensemble averaging, for $\rho>\rho_{c,\mathrm{BTW}}$, only configurations leading to $Q\neq 0$.

rations, above which the state with $Q\approx 0.17$ is accessible by the dynamics. In the neighborhood of the critical value the function $Q(\rho)$ is well fitted by a sigmoidal leading to the estimation of the critical density of the SMBTW FES $\rho_{c,\mathrm{SMBTW}}=2.0495\pm 0.0072$. The set of configurations leading to the nonvanishing value of Q can be used to present the phase diagram of the model in the (Q,ρ) space. As we show in Fig. 2(a), at $\rho=\rho_{c,\mathrm{SMBTW}}$, an abrupt jump in Q, possessing the caharacteristics of a first-order phase transition, occurs. The plot $Q=Q(\rho)$ for the common BTW model is shown in Fig. 2(b). As mentioned in the literature [14], the nature of the phase transition in this case is not clear due to the devil's staircase form of the function $Q(\rho)$ [14,15]. Our analysis, however, supports the scenario of a continuous transition in the BTW FES model. This is due to the fact that the spectrum of the stationary states $P(Q)$ accessible by the dynamics in the asymptotic limit possesses no energy gap, in contrast to the SMBTW model.

In order to support further our conjecture concerning the abrupt behavior in the SMBTW FES model, in contrast with a smooth transition of the usual BTW FES model, one has to calculate the dependence of the gap of Q on the lattice size L. We have performed a calculation of the gap in the SMBTW model for five different lattice sizes: 80, 120, 200, 600, and 1000. For lattices with size greater than $L=120$, the value of the gap is almost constant: $Q\approx 0.169$. The results of this calculation are shown with crosses in Fig. 3. A solid line at $Q=0.169$ is drawn to guide the eye. Additionally, we have calculated the asymptotic value Q for the BTW FES model using the same lattice sizes as for the SMBTW case. The corresponding results are presented in Fig. 3 with full circles. It is clearly seen that the asymptotic value Q in this case

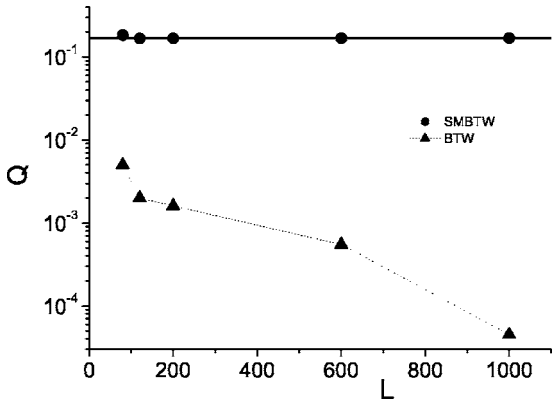

FIG. 3. The gap in Q as a function of the lattice size L for the SMBTW (full circles) as well as the BTW (full triangles) FES model (semilog plot). The solid line at $Q=0.169$ is used to guide the eye.

tends rapidly to zero with increasing lattice size. Thus, in the thermodynamic limit, the gap in the SMBTW remains finite and large, characteristic for a first-order transition, while in the BTW case no such gap occurs leading to a continuous transition.

To illustrate this property more transparently we present in Fig. 4 the distribution $P(Q)$ for the two models. To allow for a comparison, as the critical density is different in the two cases, we calculate $P(Q)$ at densities $\rho_i=\rho_{c,i}+\delta\rho$ $(i=1,2)$ using the same value $\delta\rho$ for the two models.

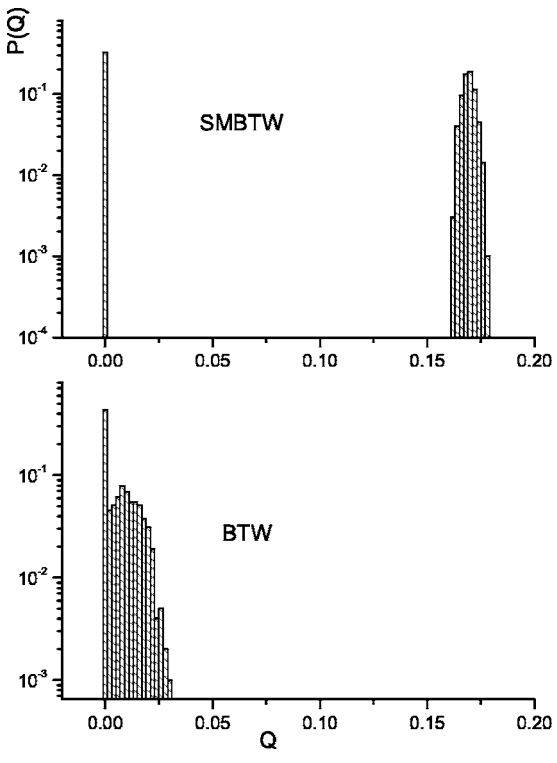

FIG. 4. The distribution $P(Q)$ for the SMBTW and BTW models at densities $\rho_1=2.060$, and $\rho_2=2.1255$, respectively ($\delta\rho=0.0105$ for both models).

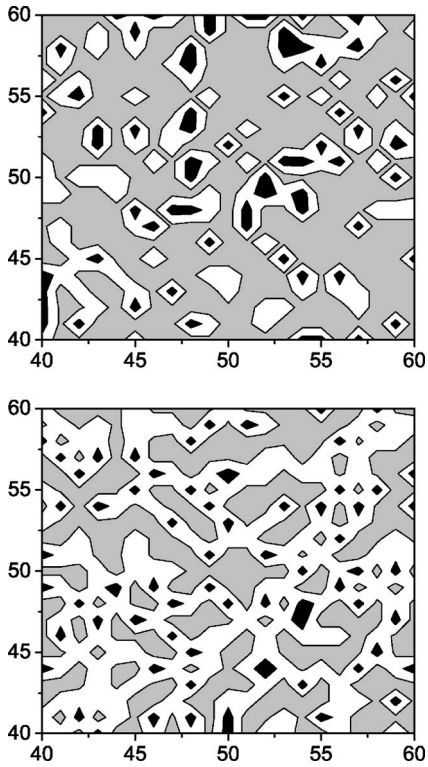

FIG. 5. The evolution of a typical configuration using the SMBTW rules for $\rho=2.060$. We present the initial state (top) as well as the resulting state for $t=5000$ (bottom). As explained in the text only a part of the lattice is displayed.

III. INTERPRETATION OF THE NUMERICAL RESULTS

In order to understand the origin of the sharp transition in the SMBTW model, we investigate the corresponding dynamics at the microscopic level. We define the single-site states of the system in terms of the possible values of the occupancies (number of grains on the site) n_s at a given time. To simplify our analysis we also include in the state $n_s=4$ the (rare) case when a site is occupied with more than four grains. For densities $\rho>\rho_c$, we observe that the single-site dynamics are characterized by an ergodic behavior: each one-site trajectory visits irregularly all the accessible states in phase space. After a characteristic time scale $t\approx 2000$ (algorithmic time in units of lattice sweeps) an invariant density is established. Having achieved this stationarity, each state is visited with almost equal probability by the dynamically evolving site. A very smooth maximum occurs for $n_s=1$ and $n_s=3$. As the number of possible states is 5 $(n_s=0,1,2,3,4)$ we expect, for a uniform invariant density and assuming that the ergodic hypothesis applies, to have a $1/5$ probability to be on the active state $(n_s=4)$, a value leading to $Q=0.2$ very close to the observed value $Q\approx 0.17$. The deviation is due to the fact that the invariant density is not exactly uniform.

It is worth exploring the global dynamics of the system as well. Therefore, we investigate the evolution of a typical configuration of the entire lattice in the critical region $\rho>\rho_c$. In Fig. 5 we present the evolution of such a configuration for $\rho=2.060$. We show the contour plots for the initial

configuration (top) as well as the evolving state at algorithmic time $t=5000$ (bottom). We recall here that stationarity is achieved already at $t=2000$. To simplify our representation without losing physical insight, we adopt the coarse-grained description of the phase space of the system used in [6,16] distinguishing among stable ($n_s<3$, gray), critical ($n_s=3$, white), and active ($n_s>3$, black) sites. For a better visualization of the details of the dynamics, it is convenient to zoom into a part of the lattice which we choose here to be the set of sites (i,j) with $i,j\in[40,60]$. Obviously, as displayed in the top plot of Fig. 5, initially the active sites form clusters that are embedded in domains consisting of critical sites.

As the dynamics evolve, the active clusters split and their gradual destruction initiates in favor of the formation of larger domains of critical sites. Before the active clusters disappear, the domains of critical sites approaching each other collide, forming larger domains of critical sites containing smaller active clusters. When the stationary state is achieved, the critical sites form large connected domains, similar to the above-threshold percolation clusters, which span the entire lattice. A dynamical equilibrium is established characterized by an irregular deformation of the critical domains as well as a chaotic motion, incorporating the splitting and recombining through collisions, of the active clusters within these domains. Thus, the critical domains construct a protecting network of communicating channels (see the bottom plot of Fig. 5) for the irregular evolution of the active clusters. It must be noted that critical clusters can also be formed in the conventional sandpile models. However, the maximal size of critical clusters in the case of the FES SMBTW model is much larger than the corresponding size of the conventional models.

The formation of this network is a process depending very sensitively on the initial conditions. Consider for example a typical configuration leading to $Q=0$ for $\rho=2.0495$. The total number of grains, for a $L=120$ lattice, in such a configuration is 25 912. Inserting one additional grain in the lattice can lead (depending on where we put it) to $Q\approx0.17$. Thus, density fluctuations of the order of 10^{-5} may lead to a tremendous change of the order parameters' values. We attribute this behavior to the first-order character of the transition in our model. Actually imposing a time ordering in the toppling rules of the system introduces an internal time scale of the order of the lattice size L. Changes in the environment of each site during time intervals of this order are felt by the corresponding site due to the dependence of the local energy current on the configuration of the environment expressed through the negative gradient constraint. This feedback mechanism creates an unstable environment, which leads to a strong sensitivity to initial conditions. The relative timescale for which this instability influences the dynamics is of the order of $1/L$ and therefore environmental influence becomes continuous in time in the limit of an infinite system. We have investigated the transition by relaxing each time one of the two constraints we have used in the SMBTW model; i.e., the negative gradient and the time ordering. In both cases the transition turns out to be continuous and the corresponding phase diagram is very close to the diagram shown in Fig. 2(b) for the BTW model. Keeping only the time-ordering constraint (SBTW model), the internal time scale has no consequence on the evolution of the system as the corresponding energy currents do not depend on the environment. Keeping only the negative gradient constraint (PMBTW model) there is no internal time and the changes in the environment occur in time scales of the order of L^2, which become less and less important as the system size increases. On the other hand, the choice of a negative gradient (instead of, for example, a positive gradient) constraint is necessary in order to reach a stationary absorbing state.

IV. CONCLUDING REMARKS

Let us now summarize briefly our results. In the present work we have introduced a sandpile model resulting from the BTW rules through the addition of two constraints: energy is transferred from an active site only to less energetic neighboring sites and the toppling takes place in an asynchronous manner. The latter means that the instants when a lattice site can topple are ordered in time. We have investigated the case without external driving. The system undergoes an abrupt (first-order) transition. Our model resembles the dynamics of activated random walkers [17] with the additional property of walking only to less occupied sites. This introduces a feedback mechanism influencing locally the energy flow on the lattice. The proposed model is minimal in the sense that both constraints are necessary in order to achieve the discontinuous change of the order parameter. Considering the one-site as well as the global dynamics of the system we were able to understand the qualitative as well as some quantitative features of the basic mechanism underlying the observed transition. Our model is capable to describe physical systems characterized by asynchronous spontaneous energy transfer. In particular the proposed model could give additional insight in prefracture processes during earthquake (EQ) events. In this case the imposed two constraints are fulfilled: (i) the stress within the focal area is transferred from regions of higher tension to regions of lower tension (negative gradient), while (ii) the microcrack transmission takes place sequentially. When the fracture of the heterogeneous environment is consummated, the remaining asperities suffer an intensive stress from their environment. A tiny fluctuation of the surrounding stress field is decisive for the final fracture of asperities and therefore the occurrence of an EQ event or not. In the former case the corresponding transition is abrupt (first order) [18,19]. Additionally, the extreme sensitivity of the proposed model on density fluctuations of relative magnitude of the order of 10^{-5} (or even less for lattices with $L>120$) suggests the possibility to design a high-efficiency sensor by appropriate realization of the proposed dynamical rules. It remains a challenging task to determine to what extent the observed behavior possesses universal features.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); P. Bak, *How Nature Works* (Copernicus, New York, 1996).
[2] G. Grinstein, in *Scale Invariance, Interfaces and Nonequilibrium Dynamics*, edited by A. McKane *et al.*, NATO Advanced Study Institute, Series B: Physics (Plenum, New York, 1995), Vol. 344.
[3] T. Hwa and M. Kardar, Phys. Rev. A **45**, 7002 (1992).
[4] D. Sornette, A. Johansen, and I. Dornic, J. Phys. I **5**, 325 (1995).
[5] A. Vespignani, S. Zapperi, and V. Loreto, Phys. Rev. Lett. **77**, 4560 (1996); J. Stat. Phys. **88**, 47 (1997).
[6] A. Vespignani and S. Zapperi, Phys. Rev. Lett. **78**, 4793 (1997); Phys. Rev. E **57**, 6345 (1998).
[7] R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **57**, 5095 (1998).
[8] M. A. Muñoz, R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **59**, 6175 (1999).
[9] A. Chessa, E. Marinari, and A. Vespignani, Phys. Rev. Lett. **80**, 4217 (1998).
[10] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[11] A. Diaz-Guilera, Europhys. Lett. **26**, 177 (1994); L. Pietronero, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. **72**, 1690 (1994); A. Ben-Hur and O. Biham, Phys. Rev. E **53**, R1317 (1996); J. Hasty and K. Wiesenfeld, J. Stat. Phys. **86**, 1179 (1997); D. Dhar, cond-mat/9909009; S. Lübeck, Int. J. Mod. Phys. B **18**, 3977 (2004).
[12] K. T. Leung, J. V. Andersen, and D. Sornette, Phys. Rev. Lett. **80**, 1916 (1998).
[13] S. Zapperi, P. Ray, H. E. Stanley, and A. Vespignani, Phys. Rev. Lett. **78**, 1408 (1997).
[14] F. Bagnoli, F. Cecconi, A. Flammini, and A. Vespignani, Europhys. Lett. **63**, 512 (2003).
[15] A. Malakis, C. Cousouris, F. K. Diakonos, and O. Biham, *Condensed Matter Theories*, Vol. 15 (Nova Science Publishers Inc., New York, 2000) p. 105; A. Malakis and F. K. Diakonos, Fractals **12**, 303 (2004).
[16] S. S. Manna, J. Phys. A **24**, L363 (1991).
[17] R. Dickman, M. A. Munoz, A. Vespignani, and S. Zapperi, Braz. J. Phys. **30**, 27 (2000).
[18] J. V. Andersen, D. Sornette, and K. Leung, Phys. Rev. Lett. **78**, 2140 (1997).
[19] Y. F. Contoyiannis, P. G. Kapiris, and K. A. Eftaxias, Phys. Rev. E **71**, 066123 (2005).